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ON THE SIMULTANEOUS COMPUTATION OF THE SECULAR AND RESONANCE EFFECTS IN THE MOTION OF CELESTIAL BODIES

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SUMMARY

The stability of the orbit of a celestial body depends predominantly upon the long period perturbations in its elements. The long period effects produced by the sun and the moon can prolong or shorten the lifetime of a satellite considerably. In every attempt to solve the problem of stability of the asteroidal ring the long period effects must be taken into account. Unfortunately, no convenient expansion of the disturbing function exists for large values of the eccentricity, large values of the inclinations, or large values of the ratio of the semimajor axes of the disturbed and disturbing body. Thus numerical integration must be used.

In the problem of determining orbital stability the disturbing function cannot be used in its standard form, because it contains the combined effects of the long and short period terms. If the short period terms are not removed, then the integration step will be too short, and the accumulation of the errors from rounding off will be too great over a long time interval. Thus, before integration all short period effects were removed from the disturbing function by a numerical process which avoids the development into series. In a previous article by the author the suggestion was made to use Halphen's method to compute the effects caused by the secular terms.

However, the author feels that the results of computation cannot be considered complete if they do not include the near resonance effects. As a result, Halphen's method is extended by inclusion of the near resonance effects caused by the commensurability of the mean motions of the distrubed and disturbing bodies. Liouville's method reduces the problem to a single harmonic analysis. The introduction of a nonsingular set of elements permits the extension of the method to the case of near circular orbits and to the case of low inclinations of the orbital planes.

CONTENTS

List of Symbols

k - the Gaussian constant

ϵ - the angle between the equatorial and the ecliptical planes 1950.0

$\mathbf{r}$ - the position vector of the disturbed body

u - the eccentric anomaly of the disturbed body

f - the true anomaly of the disturbed body

$l, \omega, \Omega, \pi, i, e, a, n$ - the standard elliptic elements of the disturbed body

$\mathbf{P}, \mathbf{Q}, \mathbf{R}$ - Gibbs' vectorial elements of the disturbed body

σ - the angular distance of the departure point in the orbit plane of the disturbed body from the ascending node

χ - the orbital true longitude of the perigee

$$\xi = e \cos \chi$$

$$\eta = e \sin \chi$$

L - the mean longitude of the disturbed body

$L_1 = l + \chi$ - the orbital mean longitude of the disturbing body

$$X = r \cos (f + \chi)$$

$$Y = r \sin (f + \chi)$$

$$p = \frac{\cos \frac{\Omega + \sigma}{2}}{\cos \frac{\Omega - \sigma}{2}} \tan \frac{i}{2}$$

$$q = \frac{\sin \frac{\Omega + \sigma}{2}}{\cos \frac{\Omega - \sigma}{2}} \tan \frac{i}{2}$$

$$s = \tan \frac{\Omega - \sigma}{2}$$

$\mathbf{r}'$ - the position vector of the disturbing body

u' - the eccentric anomaly of the disturbing body

$l', \omega', \Omega', \pi', i', e', a', n'$ - the standard elliptic elements of the disturbing body

$\mathbf{P}', \mathbf{Q}', \mathbf{R}'$ - Gibbs' vectorial elements of the disturbing body

ρ - the distance between the disturbed and the disturbing body

S - the component of the disturbing force in the direction of the radius vector of the disturbed body

T - the component of the disturbing force lying in the orbital plane and normal to the radius vector

W - the component of the disturbing force normal to the orbital plane.

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INTRODUCTION

The problem of determining the long period effects in the motions of celestial bodies is a central problem in the determination of orbital stability over a long time interval. Two types of long period perturbations are of primary importance in treating the problem of stability:

1. The effects produced by the secular terms in the disturbing function. The arguments of the secular terms contain neither the mean anomaly of the disturbing body nor the mean anomaly of the disturbed body.

2. The effects produced by the commensurability of mean motions. If the ratio of the mean motions of the disturbed and the disturbing bodies can be approximated by a rational number, then a small divisor will appear when the integration is performed analytically. The terms producing such small divisors are named "the critical terms."

The long period effects as well as their interactions can be determined analytically, but only under the restrictions that the eccentricities or inclinations are small and that only one critical term is present.

The secular effects have been treated numerically with Halphen-Goriachev theory (References 1-4). The only restriction imposed on the elements by this method is that a very close approach between the disturbed and the disturbing bodies cannot be considered. The secular effects of the first and second order were computed by means of numerical integration over an interval of 10-20 years for artificial satellites, and over an interval of many thousands of years for minor planets. Several unexpected features and large variations of the elements affecting their stability have been discovered (References 4 and 5).

However, the scheme previously used becomes incomplete if there are critical terms in the development of the disturbing function. The effects produced by the critical terms in the elements might become comparable with those produced by the secular terms, and in addition, the problem of the mutual influence of terms of both types arises.

COMPUTATION OF THE SYSTEM OF FORMULAS FOR SECULAR AND RESONANCE EFFECTS

The system of formulas given in this article is based on the application of the method of Liouville (Reference 6) and on Halphen's method. It takes into account the secular as well as the critical effects. The equations for variations of the elements are also given in the form which includes the cases of small eccentricity and small inclination. Let $F(u, u')$ be a function of the form

$$F(u, u') = \sum_{j=0}^{+\infty} \sum_{j'=-\infty}^{+\infty} \left[A_{j,j'} \cos(jl - j'l') + B_{j,j'} \sin(jl - j'l')\right], \tag{1}$$

where

$$l = u - e \sin u ,$$

$$l' = u' - e' \sin u' ,$$

and the coefficients $A_{j,j'}$, $B_{j,j'}$ are functions of all the remaining elements. If it is assumed that

$$\frac{n'}{n} \approx \frac{i}{i'}$$

where i and i' are relative primes, then the critical arguments have the form

$$k(il - i'l') .$$

By using the Liouville substitution (Reference 6),

$$l = i'\sigma' ,$$

$$l' = i\sigma' - \frac{\theta}{i'} ,$$

Equation 1 can be written in the form:

$$F(u, u') = \sum_{j=0}^{+\infty} \sum_{j'=-\infty}^{+\infty} \left\{A_{j,j'} \cos\left[(ji' - j'i)\sigma' + \frac{j'}{i'}\theta\right] + B_{j,j'} \sin\left[(ji' - j'i)\sigma' + \frac{j'}{i'}\theta\right]\right\}. \tag{2}$$

The angle σ' is not contained in the argument if

$$ji' - j'i = 0 ,$$

or, consequently, if

$$\left.\begin{aligned} j &= pi \,, \\ j' &= pi' \,, \end{aligned}\right\} \tag{3}$$

where $p = 0, 1, 2, \cdots$. Let us designate the combination of all terms in $F(u, u')$ which do not contain σ' by $\Psi(\theta)$:

$$\Psi(\theta) = \sum_{p=0}^{+\infty} \left(A_{pi,pi'} \cos p\theta + B_{pi,pi'} \sin p\theta \right) .$$

By taking Equation 3 into account, it may be deduced from Equation 2 that

$$\begin{aligned} \Psi(\theta) &= \frac{1}{2\pi} \int_0^{2\pi} F(u, u')\, d\sigma' \\ &= \sum_{p=0}^{+\infty} \left(A_{pi,pi'} \cos p\theta + B_{pi,pi'} \sin p\theta \right) . \end{aligned} \tag{4}$$

Putting $\theta = 0, \pi, \pi/2, 3\pi/2, \pi/4$, in Equation 4 yields:

$$\begin{aligned} \Psi(0) &= A_{0,0} + A_{i,i'} + A_{2i,2i'} + A_{3i,3i'} + \cdots , \\ \Psi(\pi) &= A_{0,0} - A_{i,i'} + A_{2i,2i'} - A_{3i,3i'} + \cdots , \\ \Psi\left(\frac{\pi}{2}\right) &= A_{0,0} + B_{i,i'} - A_{2i,2i'} - B_{3i,3i'} + \cdots , \\ \Psi\left(\frac{3\pi}{2}\right) &= A_{0,0} - B_{i,i'} - A_{2i,2i'} + B_{3i,3i'} + \cdots , \\ \Psi\left(\frac{\pi}{4}\right) &= A_{0,0} + \frac{\sqrt{2}}{2} \left(A_{i,i'} + B_{i,i'} \right) + B_{2i,2i'} + \cdots . \end{aligned}$$

It is rarely necessary to go beyond the critical term with the argument $2il - 2i'l'$. Thus, the coefficients $A_{3i,3i'}$, $B_{3i,3i'}$, etc. can be neglected and a set of simple formulas results:

$$A_{i,i'} = \frac{\Psi(0) - \Psi(\pi)}{2} , \tag{5}$$

$$B_{i,i'} = \frac{\Psi\left(\frac{\pi}{2}\right) - \Psi\left(\frac{3\pi}{2}\right)}{2} , \tag{6}$$

$$A_{2i,2i'} = \frac{\Psi(0) - \Psi\left(\frac{\pi}{2}\right) + \Psi(\pi) - \Psi\left(\frac{3\pi}{2}\right)}{4}, \tag{7}$$

$$B_{2i,2i'} = \Psi\left(\frac{\pi}{4}\right) - \frac{\sqrt{2}+1}{4}\left[\Psi(0) + \Psi\left(\frac{\pi}{2}\right)\right] + \frac{\sqrt{2}-1}{4}\left[\Psi(\pi) + \Psi\left(\frac{3\pi}{2}\right)\right] . \tag{8}$$

In the case of a large or moderate eccentricity set

$$u = i'\phi ;$$

then

$$l' = u' - e'\sin u'$$

$$= i\phi - \frac{ie}{i'} \sin i'\phi - \frac{\theta}{i'} , \tag{9}$$

and Equation 4 takes a form more convenient for numerical computations:

$$\Psi(\theta) = \frac{1}{2\pi}\int_0^{2\pi} F(u, u') \frac{r}{a}\, d\phi . \tag{10}$$

This last integral is computed as a simple arithmetical mean of the integrands of equidistant values of the angle ϕ. Now a collection of formulas can be established for the actual computation of the effect of resonance in the elements on the basis of the Liouville equations (Equations 5-8). Setting

$$A_1(\alpha) = \begin{bmatrix} +1 & 0 & 0 \\ 0 & +\cos\alpha & -\sin\alpha \\ 0 & +\sin\alpha & +\cos\alpha \end{bmatrix} ,$$

$$A_3(\alpha) = \begin{bmatrix} +\cos\alpha & -\sin\alpha & 0 \\ +\sin\alpha & +\cos\alpha & 0 \\ 0 & 0 & +1 \end{bmatrix} ,$$

we can deduce the Gibbs vectorial elements **P**, **Q**, **R** and **P**$'$, **Q**$'$, **R**$'$ by means of the formulas:

$$[\mathbf{P}, \mathbf{Q}, \mathbf{R}] = A_1(\epsilon) \cdot A_3(\Omega) \cdot A_1(i) \cdot A_3(\omega) ,$$

$$[\mathbf{P}', \mathbf{Q}', \mathbf{R}'] = A_1(\epsilon) \cdot A_3(\Omega') \cdot A_1(i') \cdot A_3(\omega') .$$

In the case of an artificial satellite the elements ω and Ω are referred to the equator and

$$[\mathbf{P}, \mathbf{Q}, \mathbf{R}] = A_3(\Omega) \cdot A_1(i) \cdot A_3(\omega) .$$

The coordinates of the disturbed body are

$$\left.\begin{aligned} x &= a P_x (\cos u - e) + a \sqrt{1-e^2}\, Q_x \sin u \ , \\ y &= a P_y (\cos u - e) + a \sqrt{1-e^2}\, Q_y \sin u \ , \\ z &= a P_z (\cos u - e) + a \sqrt{1-e^2}\, Q_z \sin u \ , \end{aligned}\right\} \tag{11}$$

and the coordinates of the disturbing body are

$$\left.\begin{aligned} x' &= a' P_x' (\cos u' - e') + a' \sqrt{1-e'^2}\, Q_x' \sin u' \ , \\ y' &= a' P_y' (\cos u' - e') + a' \sqrt{1-e'^2}\, Q_y' \sin u' \ , \\ z' &= a' P_z' (\cos u' - e') + a' \sqrt{1-e'^2}\, Q_z' \sin u' \ . \end{aligned}\right\} \tag{12}$$

The square of the mutual distance is

$$\rho^2 = r^2 + r'^2 - 2\mathbf{r} \cdot \mathbf{r}' \ . \tag{13}$$

Let the critical argument be

$$\theta = i l - i' l' \ .$$

Set

$$u = i' \phi \ ,$$

$$u' - e' \sin u' = i\phi - \frac{\theta}{i'} - \frac{ie}{i'} \sin i' \phi \ .$$

The radial component S, the orthogonal component T, and the normal component W of the disturbing force can be considered functions of the angle ϕ and the critical argument θ. Thus

$$rS = km' \left(\frac{1}{\rho^3} - \frac{1}{r'^3} \right) \mathbf{r} \cdot \mathbf{r}' - km' \frac{r^2}{\rho^3} = rS(\phi, \theta) \ , \tag{14}$$

$$rT = km' \left(\frac{1}{\rho^3} - \frac{1}{r'^3} \right) (\mathbf{R} \cdot \mathbf{r} \times \mathbf{r}') = rT(\phi, \theta) \ , \tag{15}$$

$$W = km' \left(\frac{1}{\rho^3} - \frac{1}{r'^3} \right) \mathbf{R} \cdot \mathbf{r}' = W(\phi, \theta) \ . \tag{16}$$

From the following standard equations for variations of the elements

$$\frac{dn}{dt} = -\frac{3k}{a^2}\left(S\frac{ae\sin f}{\sqrt{1-e^2}} + T\frac{a^2}{r}\sqrt{1-e^2}\right) ,$$

$$\frac{de}{dt} = \frac{na\left(1-e^2\right)}{ke}\left(S\frac{ae\sin f}{\sqrt{1-e^2}} + T\frac{a^2}{r}\sqrt{1-e^2}\right) - \frac{na\sqrt{1-e^2}}{ke}Tr ,$$

$$\frac{d\pi}{dt} = \frac{na\sqrt{1-e^2}}{ke}\left[-Sa\cos f + Ta\left(1+\frac{1}{1-e^2}\frac{r}{a}\right)\sin f\right] + 2\sin^2\frac{i}{2}\frac{d\Omega}{dt} ,$$

$$\frac{dL}{dt} = -\frac{2nar}{k}S + \left(1-\sqrt{1-e^2}\right)\frac{d\pi}{dt} + 2\sqrt{1-e^2}\sin^2\frac{i}{2}\frac{d\Omega}{dt} ,$$

$$\sin i\,\frac{d\Omega}{dt} = \frac{nar}{k\sqrt{1-e^2}}W\sin(f+\omega) ,$$

$$\frac{di}{dt} = \frac{nar}{k\sqrt{1-e^2}}W\cos(f+\omega) ,$$

the short period and the secular terms may be eliminated by using the process described above, and the transformed equations will contain only the effect of the critical terms. By designating

$$A_{i,i'} = K_1 , \qquad B_{i,i'} = K_2 , \qquad A_{2i,2i'} = K_3 , \qquad B_{2i,2i'} = K_4 ,$$

the derivative of an element E becomes

$$\frac{dE}{dt} = K_1\cos\theta + K_2\sin\theta + K_3\cos 2\theta + K_4\sin 2\theta . \tag{17}$$

Let us introduce the following notations:

$$\frac{f(\phi, 0) - f(\phi, \pi)}{2} = f_1(\phi) ,$$

$$\frac{f\left(\phi, \frac{\pi}{2}\right) - f\left(\phi, \frac{3\pi}{2}\right)}{2} = f_2(\phi) ,$$

$$\frac{f(\phi, 0) - f\left(\phi, \frac{\pi}{2}\right) + f(\phi, \pi) - f\left(\phi, \frac{3\pi}{2}\right)}{4} = f_3(\phi) ,$$

$$f\left(\phi, \frac{\pi}{4}\right) - \frac{\sqrt{2}+1}{4}\left[f(\phi, 0) + f\left(\phi, \frac{\pi}{2}\right)\right] + \frac{\sqrt{2}-1}{4}\left[f(\phi, \pi) + f\left(\phi, \frac{3\pi}{2}\right)\right] = f_4(\phi) .$$

These values of the coefficients K_j $(j = 1, 2, 3, 4)$ are deduced:

in dn/dt

$$K_j^{(n)} = -\frac{3k}{2\pi a}\int_0^{2\pi}\left(S_j\, e\sin u + T_j\sqrt{1-e^2}\right)d\phi ; \tag{18}$$

in de/dt

$$K_j^{(e)} = \frac{\sqrt{a(1-e^2)}}{2\pi}\int_0^{2\pi}\left[\sqrt{1-e^2}\;S_j\sin u + T_j\left(-\frac{3}{2}e + 2\cos u - \frac{1}{2}e\cos 2u\right)\right]d\phi\ ; \tag{19}$$

in $d\pi/dt$

$$K_j^{(\pi)} = \frac{\sqrt{a(1-e^2)}}{2\pi e}\int_0^{2\pi}\left[-S_j(\cos u - e) + T_j\left(1+\frac{r}{a}\frac{1}{1-e^2}\right)\sqrt{1-e^2}\;\sin u\right]d\phi + 2\sin^2\frac{i}{2}\,K_j^{(\Omega)}\ ; \tag{20}$$

in dL/dt

$$K_j^{(L)} = \frac{e^2}{1+\sqrt{1-e^2}}K_j^{(\pi)} + \sqrt{1-e^2}\;K_j^{(\Omega)}\,2\sin^2\frac{i}{2} - \frac{2\sqrt{a}}{2\pi}\int_0^{2\pi}S_j\frac{r^2}{a^2}\,d\phi\ ; \tag{21}$$

in $d\Omega/dt$

$$K_j^{(\Omega)} = \frac{1}{2\pi\sin i}\sqrt{\frac{a}{1-e^2}}\int_0^{2\pi}W_j\frac{r}{a}\left[(\cos u - e)\sin\omega + \sqrt{1-e^2}\;\sin u\cos\omega\right]d\phi\ ; \tag{22}$$

in di/dt

$$K_j^{(i)} = \frac{1}{2\pi}\sqrt{\frac{a}{1-e^2}}\int_0^{2\pi}W_j\frac{r}{a}\left[(\cos u - e)\cos\omega - \sqrt{1-e^2}\;\sin u\sin\omega\right]d\phi\ . \tag{23}$$

It is interesting to note that Equations 18-23 are similar in form to the equations for the computation of secular effects. The cases of small inclination and small eccentricity deserve special attention. Equations 20 and 22 contain a "small divisor" if i or e is very small. Instead of the standard orbital elements, a nonsingular set of elements can be introduced. The set of elements

$$\left.\begin{aligned} p &= \frac{\cos\frac{\Omega+\sigma}{2}}{\cos\frac{\Omega-\sigma}{2}}\tan\frac{i}{2}\ , \\ q &= \frac{\sin\frac{\Omega+\sigma}{2}}{\cos\frac{\Omega-\sigma}{2}}\tan\frac{i}{2}\ , \\ s &= \tan\frac{\Omega-\sigma}{2}\ , \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned} \xi &= e\cos\chi\ , \\ \eta &= e\sin\chi\ , \end{aligned}\right\} \tag{25}$$

determines the position of the osculating ellipse without the introduction of a singularity for $i = 0$ or $e = 0$. The use of Equations 25 for the computation of perturbations was suggested by the author in his article on Strömgren's perturbations (Reference 7). In fact, p, q, and s are the components of Gibbs' rotation vector (Reference 8). From the equations

$$\sin i \frac{d\Omega}{dt} = \frac{rW}{\sqrt{a\left(1-e^2\right)}} \sin(f+\omega) ,$$

$$\frac{di}{dt} = \frac{rW}{\sqrt{a\left(1-e^2\right)}} \cos(f+\omega) ,$$

$$\frac{d\sigma}{dt} = \cos i \frac{d\Omega}{dt} ,$$

it is deduced that

$$\begin{aligned}
\frac{dp}{dt} &= \frac{W}{2\sqrt{a\left(1-e^2\right)}} \left[\left(1+p^2\right) X + (pq-s) Y\right] , \\
\frac{dq}{dt} &= \frac{W}{2\sqrt{a\left(1-e^2\right)}} \left[(pq+s) X + \left(1+q^2\right) Y\right] , \\
\frac{ds}{dt} &= \frac{W}{2\sqrt{a\left(1-e^2\right)}} \left[(ps-q) X + (qs+p) Y\right] .
\end{aligned} \tag{26}$$

It can be seen that no small divisor is present, even if the inclination is small. X and Y are the coordinates of the disturbing body in an ideal system of coordinates with the X- and Y-axes in the orbital plane. They can be written in the form:

$$\begin{aligned}
X &= r\cos(f+\chi) \\
&= a\left[\left(\frac{1+\sqrt{1-e^2}}{2} + \frac{\xi^2-\eta^2}{2}\,\frac{1}{1+\sqrt{1-e^2}}\right)\cos\lambda + \frac{\xi\eta}{1+\sqrt{1-e^2}}\sin\lambda - \xi\right] ,
\end{aligned} \tag{27}$$

$$\begin{aligned}
Y &= r\sin(f+\chi) \\
&= a\left[\left(\frac{1+\sqrt{1-e^2}}{2} - \frac{\xi^2-\eta^2}{2}\,\frac{1}{1+\sqrt{1-e^2}}\right)\sin\lambda + \frac{\xi\eta}{1+\sqrt{1-e^2}}\cos\lambda - \eta\right] ,
\end{aligned} \tag{28}$$

where λ is the eccentric orbital longitude,

$$\lambda = u + \chi . \tag{29}$$

Also (Reference 9)

$$\frac{d\xi}{dt} = \sqrt{a\left(1-e^2\right)} \left[S\sin(f+\chi) + T\left(1+\frac{r}{a}\,\frac{1}{1-e^2}\right)\cos(f+\chi) + \frac{r}{a}\,\frac{1}{1-e^2}\,T\xi\right] , \tag{30}$$

$$\frac{d\eta}{dt} = \sqrt{a(1-e^2)}\left[-S\cos(f+\chi) + T\left(1+\frac{r}{a}\frac{1}{1-e^2}\right)\sin(f+\chi) + \frac{r}{a}\frac{1}{1-e^2}T\eta\right]. \tag{31}$$

We write the critical argument in the form:

$$\alpha = iL_1 - i'l'.$$

Then the Liouville substitution becomes

$$L_1 = i'\sigma',$$

$$l' = i\sigma' - \frac{\alpha}{i'}.$$

Considering the disturbing force as a function of σ' and α, set

$$\frac{f(\sigma', 0) - f(\sigma', \pi)}{2} = f_1(\sigma'), \text{ etc.}$$

From Equations 26, 30, and 31 the following expressions are deduced for the coefficients of the critical terms:

$$K_j^{(p)} = \frac{1}{4\pi\sqrt{a(1-e^2)}}\int_0^{2\pi} W_j\left[(1+p^2)X + (pq-s)Y\right]d\sigma', \tag{32}$$

$$K_j^{(q)} = \frac{1}{4\pi\sqrt{a(1-e^2)}}\int_0^{2\pi} W_j\left[(pq+s)X + (1+q^2)Y\right]d\sigma', \tag{33}$$

$$K_j^{(s)} = \frac{1}{4\pi\sqrt{a(1-e^2)}}\int_0^{2\pi} W_j\left[(ps-q)X + (qs+p)Y\right]d\sigma', \tag{34}$$

$$K_j^{(\xi)} = \frac{1}{2\pi}\sqrt{\frac{1-e^2}{a}}\int_0^{2\pi}\left[+S_jY + T_j\left(1+\frac{r}{a}\frac{1}{1-e^2}\right)X + \frac{r^2}{a^2}\frac{1}{1-e^2}T_j\xi\right]\frac{a}{r}d\sigma', \tag{35}$$

$$K_j^{(\eta)} = \frac{1}{2\pi}\sqrt{\frac{1-e^2}{a}}\int_0^{2\pi}\left[-S_jX + T_j\left(1+\frac{r}{a}\frac{1}{1-e^2}\right)Y + \frac{r^2}{a^2}\frac{1}{1-e^2}T_j\xi\right]\frac{a}{r}d\sigma'. \tag{36}$$

The expression for the position vector of the disturbed body to be used in the computations of the integrands is:

$$\mathbf{r} = \frac{A_3(\epsilon)\cdot\Gamma}{1+p^2+q^2+s^2}\begin{bmatrix}X\\Y\\0\end{bmatrix}, \tag{37}$$

where the elements γ_{ij} (i is the row index and j is the column index) of the matrix Γ are determined from the formulas (Reference 10):

$$\left.\begin{array}{lll} \gamma_{11} = +p^2 - q^2 - s^2 + 1 \,, & \gamma_{21} = +2(s+pq) \,, & \gamma_{31} = +2(sp-q) \,, \\ \gamma_{12} = -2(s-pq) \,, & \gamma_{22} = -p^2 + q^2 - s^2 + 1 \,, & \gamma_{32} = +2(p+sq) \,, \\ \gamma_{13} = +2(q+ps) \,, & \gamma_{23} = -2(p-sq) \,, & \gamma_{33} = -p^2 - q^2 + s^2 - 1 \,. \end{array}\right\} \tag{38}$$

X and Y are given by Equations 27-29. Kepler's equation is replaced by

$$\lambda - \xi \sin\lambda + \eta \cos\lambda = L_1 \tag{39}$$

in a way similar to Herget's method of writing Kepler's equation for small eccentricities (Reference 11). Similar systems of equations can be established for numerical integration of the secular effects for the case of a small eccentricity or a small inclination.

The effects of the secular and critical terms must be integrated together. Again Halphen's method may be used for the computations of the secular effects. A complete collection of formulas is given in the author's previous article (Reference 3). However, if i or e, or both, are small, the system requires some modification.

For the values, say, $\lambda = 0°, 10°, \cdots, 350°$, compute:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1 + p^2 + q^2 + s^2} \begin{bmatrix} P_x' & P_y' & P_z' \\ Q_x' & Q_y' & Q_z' \\ R_x' & R_y' & R_z' \end{bmatrix} \cdot \Gamma \cdot \begin{bmatrix} X \\ Y \\ 0 \end{bmatrix} \,. \tag{40}$$

X and Y are defined by Equations 27 and 28. Kepler's equation is taken in the form of Equation 39. Then, as in the previous article, put

$$\alpha = x' + e'a \,,$$

$$\beta = y \,,$$

$$\gamma = z \,.$$

Equations 4-9 of Reference 3 remain unchanged, but Equation 10, p. 121, for the variations of the elements must be replaced by:

$$\frac{dn}{dt} = -\frac{3k}{a}\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{S_0(Y\xi - X\eta)}{\sqrt{1-e^2}} + T_0\sqrt{1-e^2}\right] d\lambda \,, \tag{41}$$

$$\frac{d\xi}{dt} = \sqrt{\frac{1-e^2}{a}}\,\frac{1}{2\pi}\int_0^{2\pi}\left[S_0 Y + T_0\left(1+\frac{r}{a}\,\frac{1}{1-e^2}\right)X + \left(\frac{r}{a}\right)^2 \frac{a}{1-e^2}\,T_0\xi\right]d\lambda\;, \tag{42}$$

$$\frac{d\eta}{dt} = \sqrt{\frac{1-e^2}{a}}\,\frac{1}{2\pi}\int_0^{2\pi}\left[-S_0 X + T_0\left(1+\frac{r}{a}\,\frac{1}{1-e^2}\right)Y + \left(\frac{r}{a}\right)^2 \frac{a}{1-e^2}\,T_0\eta\right]d\lambda\;, \tag{43}$$

$$\frac{dp}{dt} = \frac{1}{2\sqrt{a\left(1-e^2\right)}}\,\frac{1}{2\pi}\int_0^{2\pi} W_0\,\frac{r}{a}\left[\left(1+p^2\right)X + (pq-s)\,Y\right]d\lambda\;, \tag{44}$$

$$\frac{dq}{dt} = -\frac{1}{2\sqrt{a\left(1-e^2\right)}}\,\frac{1}{2\pi}\int_0^{2\pi} W_0\,\frac{r}{a}\left[(pq+s)\,X + \left(1+q^2\right)Y\right]d\lambda\;, \tag{45}$$

$$\frac{ds}{dt} = \frac{1}{2\sqrt{a\left(1-e^2\right)}}\,\frac{1}{2\pi}\int_0^{2\pi} W_0\,\frac{r}{a}\left[(ps-q)\,X + (qs+p)\,Y\right]d\lambda\;, \tag{46}$$

$$\frac{dL_1}{dt} = -\frac{2}{\sqrt{a}}\,\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 S_0\,d\lambda + \frac{\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}}{1+\sqrt{1-e^2}}\;. \tag{47}$$

CONCLUSIONS

The combined effects of the secular and critical terms have a deciding influence in determining the orbital stability over a long time interval. Some approximate schemes based on the process of averaging lead to conclusions concerning the stability of the ring of minor planets which definitely require a further check. The proposed scheme will at least give a more accurate answer over an interval of some thousands of years. Our knowledge about the stability of the orbits of artificial satellites is still incomplete. The described method will be programmed in order to investigate these problems. The application of the method of secular effects leads to some conclusions which were not anticipated. It remains to be seen what interesting conclusions will follow from the superposition of all long period terms and from the computations of their direct actions, as well as their interactions, in the problems of orbital stability of the minor planets and artificial satellites.

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